Quasi-Polynomial Algebras (Commutative Algebra and Algebraic Geometry)

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QUASI-POLYNOMIAL ALGEBRAS

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Introduction.

Let R be a commutative ring, and let $R^{[n]}$ be a polynomial ring $R[X_1,\ldots,X_n]$. For a prime ideal $\wp$ of R, we denote by $j(\wp)$ the integral closure of $R/\wp$ in its quotient field $k(\wp) = R_\wp/\wp R_\wp$. There are natural injections $R/\wp \hookrightarrow j(\wp) \hookrightarrow k(\wp)$, and hence $j(\wp)$ may be considered as an R-subalgebra of $k(\wp)$.

Definition. An R-algebra A is called a quasi-polynomial R-algebra in n-variables if the base extension $j(\wp) \otimes_R A$ is $j(\wp)$-isomorphic to $j(\wp)^{[n]}$ for each prime ideal $\wp$ of R.

Of course the polynomial ring $R^{[n]}$ is a trivial example of quasi-polynomial algebras. Conversely if R is an integrally closed local domain, then any quasi-polynomial R-algebra is a polynomial ring over R. But there is a non-trivial example of quasi-polynomial R-algebras in general as follows:

Example. (a): Let k be a field of positive characteristic p, and let $R = k[Y^2,Y^3] \subset k[Y] \cong_k k^{[1]}$. If we set $A = R[Z^p, Z+YZ^p] \subset k[Y,Z] \cong_k k^{[2]}$, then $A^{[1]} \cong_R R^{[2]}$ and $A \not\cong_R R^{[1]}$. Moreover A is a quasi-polynomial R-algebra in one-variable ([2],[12]).

(b): Similarly if we set $R = \mathbb{Z}[2\sqrt{2}]$ and $A = R[Z^2, Z+\sqrt{2}Z^2]$, then

$A^{[1]} \cong_R R^{[2]}$ and $A \ncong_R R^{[1]}$. This is another non-trivial example of quasi-polynomial R-algebras ([3]).

The pourpose of this article is to discuss some topics related to quasi-polynomial algebras. For the detail we refer to [5] and [6].

1. Generalities.

We will need some definitions from [7] and [9].

Definition. An R-algebra A is called weakly projective if A is a retract of a polynomial ring $R^{[n]}$, i.e., there is a pair of R-homomorphisms

$$A \xrightarrow{g} R^{[n]} \xrightarrow{f} A$$

such that $f \circ g = id_A$, the identity map on A. An R-algebra A is called invertible if there is an R-algebra B such that

$$A \otimes_R B \cong_R R^{[n]}.$$

An invertible algebra is weakly projective but the converse does not necessarily hold.

Definition. An R-algebra A is called strongly projective if there is a projective A-module M such that the symmetric algebra $S_A(M)$ is R-isomorphic to $R^{[n]}$.

In this case the symmetric algebra $S_A(M)$ is considered as an R-algebra by natural homomorphisms $R \longrightarrow A \longrightarrow S_A(M)$. These three types of algebras are very closely related to quasi-polynomial algebras. It is easy to see that the symmetric algebra $S_R(N)$ of a finitely-generated projective R-module N is a typical example of these three algebras.

Theorem 1.1. Let R be a noetherian ring, and let A be a finitely-generated, flat, quasi-polynomial R-algebra such that the differential A-module $\Omega_R(A)$ is projective. Then A is a strongly projective R-algebra.

In the case that R is reduced, we can omit the assumption of $\Omega_R(A)$ to be projective from Theorem 1.1. Thus we have the following corollary of Theorem 1.1.

Corollary 1.2. Let R be a reduced noetherian ring, and let A be a finitely-generated, flat, quasi-polynomial R-algebra. Then A is a strongly projective R-algebra.

In the following we consider some applications of quasi-polynomial algebras.

2. Algebras stably equivalent to $R^{[1]}$.

Definition. An R-algebra A is called stably equivalent to an R-algebra B if $A^{[n]} \cong_R B^{[n]}$.

Lemma 2.1. Let A be a strongly projective R-algebra. Then A is stably equivalent to $R^{[n]}$ if and only if $\Omega_R(A)$ is stably equivalent (as an A-module) to a free A-module A^n of rank n, i.e., $\Omega_R(A) \oplus A^r \cong A^{n+r}$.

Proposition 2.2. Let R be a noetherian ring, and let A be a finitely-generated, flat, quasi-polynomial R-algebra. Then A is stably equivalent to $R^{[n]}$ if and only if $\Omega_R(A)$ is stably equivalent to A^n.

Proposition 2.3. Let R be a noetherian ring, and let A be an R-algebra. Then the following two conditions are equivalent:

(i) A is stably equivalent to $R^{[1]}$.

(ii) A is a quasi-polynomial R-algebra in one variable such that $\Omega_R(A)$ is free.

Definition. We say that an R-algebra A is R-invariant if any R-algebra B stably equivalent to A is R-isomorphic to A ([1]).

Thus under the assumption that $R^{[1]}$ is R-invariant a finitely-generated, flat, quasi-polynomial R-algebra A in one-variable is R-isomorphic to $R^{[1]}$ if and only if $\Omega_R(A)$ is free. As shown in the introduction, $R^{[1]}$ is not always R-invariant. We give a necessary and sufficient condition on R for $R^{[1]}$ to be R-invariant. (See [2],[3],[5],[12])

Definition. Let R be a subring of a reduced ring S. We say that an element $a \in S$ is of F-type over R if a^2, a^2, $na \in R$ for some positive integer n. If each element $a \in S$ of F-type over R is contained in R, then R is saied to be F-closed in S. If a reduced ring R is F-closed in any reduced ring S containing R as a subring, then R is called an F-ring ([3]).

A reduced ring containing a field of characteristic zero is a typical example of F-rings. A semi-normal ring (i.e., a reduced ring R such that the Picard group $\mathrm{Pic}(R)$ is canonically isomorphic to $\mathrm{Pic}(R^{[n]})$ for any $n > 0$) is another example of F-rings ([12]). On the other hand $k[Y^2, Y^3]$ and $\mathbb{Z}[2\sqrt{2}]$ are the examples of non-F-rings, where k is a field of positive characteristic p.

Theorem 2.4. $R^{[1]}$ is R-invariant if and only if $R_{red} = R/\sqrt{(0)}$ is an F-ring, where $\sqrt{(0)}$ denotes the nil-radical of R.

Let R be a reduced ring with a finite-number of minimal prime ideals. The F-closure $F(R)$ of R is defined as the intersection $\bigcap_{\lambda} R_{\lambda}$ of all F-rings R_{λ} such that $R \subset R_{\lambda} \subset K$, where K is the total quotient ring of R. We note that R is an F-ring if and only if $F(R) = R$. If R is of prime characteristic p, then for any element $a \in F(R)$ there is a positive integer e such that $a^{p^e} \in R$. (The integer e may depend on a)

Theorem 2.5. Let R be a reduced noetherian ring of prime characteristic p, and let A be an R-algebra. Then the following two conditions are equivalent:

(i) A is stably equivalent to $R^{[1]}$.

(ii) $A \cong_R R[Z^{p^e}, Z+a_1Z^p+\dots+a_rZ^{rp}]$,

where $a_i \in F(R)$ such that $a_i^{p^e} \in R$.

Theorem 2.6. Let R be a ring containing an infinite field k. Let A and B be any pair of R-algebras stably equivalent to $R^{[1]}$. Then $A \otimes_R B \cong_R R^{[2]}$.

When R does not contain a field, there is a counterexample of Theorem 2.6 as follows:

Proposition 2.7. Let R be an integral domain with the quotient field K. Suppose there is an element $t \in K$ such that $t^rR[t] \subset R$ and $4t^2 \notin 2R + t^rR[t]$ for some positive integer r. We can choose an element $F \in R[t,Z]$, so that $F+tF^2 \equiv Z \pmod{t^r}$. If we set an R-algebra $A = R[F] + t^rR[t,Z]$, then $A^{[1]} \cong_R R^{[2]}$ and $A \otimes_R A \not\cong R^{[2]}$.

Example. Let $R = \mathbb{Z}[2t]$ and $r = 9$, where $t^3 = 2$. Then $R = \mathbb{Z} + \mathbb{Z}\,2t + \mathbb{Z}\,4t^2$ and $R[t] = Z + \mathbb{Z}\,t + \mathbb{Z}\,t^2$. Thus $2t \in R$ and $t^9R[t] = 8R[t] \subset 2R$. If $4t^2 \in 2R + t^9R[t] \subset 2R$, then $4t^2 \in \mathbb{Z}8t^2$. This shows that $1 \in 2\mathbb{Z}$, which is a contradiction. Therefore $4t^2 \notin 2R + t^9R[t]$. Thus $R = \mathbb{Z}[2t]$ satisfies the condition in Proposition 2.7.

Let k be an algebraically closed field of characteristic $p \geq 0$, and let R be a one dimensional affine k-domain. Then R is k-invariant ([1]). Now we consider the k-invariance of $A = R^{[1]}$.

If $R \cong k^{[1]}$, i.e., $A \cong k^{[2]}$, then A is k-invariant ([10],[13]). On the other hand, if $R \not\cong k^{[1]}$, then any k-isomorphism $f: R^{[n+1]} \to B^{[n]}$ implies $f(R) \subset B$, where B is an affine k-domain ([1],[8]). Therefore we have the following:

Theorem 2.8. Let k be an algebraically closed field of characteristic $p \geq 0$, and let R be a one dimensional affine k-domain. Suppose B is a k-algebra such that $R^{[n+1]} \cong_k B^{[n]}$.

(i) If $p = 0$, then $B \cong_k R^{[1]}$.

(ii) If $p > 0$, then $B^{[1]} \cong_k R^{[2]}$ and $B \cong_k R[Z^{p^e}, Z+a_1 Z^p + \ldots + a_r Z^{rp}]$, where $a_i \in F(R)$ such that $a_i^{p^e} \in R$. In particular $R^{[1]}$ is k-invariant if and only if R is an F-ring.

Example. Let k be a field of characteristic zero. Let us set

$$A = k[Y, Z+Z^3, (Y-1)(Y-2)Z, (Y-1)(Y-2)Z^2]$$

and

$$B = k[Y, YZ+Y^3Z^3, (Y-1)(Y-2)Z, (Y-1)(Y-2)Z^2].$$

Then $A^{[1]} \cong_k B^{[1]}$ and $A \not\cong_k B$. Thus A is not k-invariant ([4]).

3. Algebras with polynomial fibres

Theorem 3.1. Let R be a noetherian ring, and let A be a finitely-generated flat R-algebra. Suppose $k(\wp) \otimes_R A \cong k(\wp)^{[1]}$ for each prime ideal $\wp$ of R. Then A is a locally quasi-polynomial R-algebra, i.e., $R_\wp \otimes_R A$ is a quasi-polynomial $R_\wp$-algebra for each prime ideal $\wp$. (cf. [7])

Corollary 3.2. Let R be a noetherian semi-normal ring, and let A be as in Theorem 3.1. Then $A \cong_R S_R(N)$ for some $N \in \mathrm{Pic}(R)$.

Corollary 3.3. Let R be a reduced noetherian ring. Then the following two conditions are equivalent:

(i) A is a finitely-generated, flat, locally, quasi-polynomial R-algebra in one variable.

(ii) A is weakly projective R-algebra such that (Krull) $\dim k(\wp) \otimes_R A = 1$ for each prime ideal $\wp$ of R.

Corollary 3.4. ([11]) Let R be a noetherian ring, and let A be an R-subalgebra of $R^{[1]}$ such that $R^{[1]}$ is f.flat over A. Then A is a locally quasi-polynomial R-algebra.

Corollary 3.5. Let R be a noetherian ring containing a field of characteristic zero, and let A be an R-algebra. Suppose there is a finitely-generated integral extension S over R such that $S \otimes_R A \cong_S S^{[1]}$. Then A is a locally quasi-polynomial R-algebra.

4. Invertible algebras.

Let A be an invertible R-algebra. It is easy to see that $\Omega_R(A)$ is a finitely-generated projective A-module. If $\Omega_R(A)$ is of rank r, then we say that an invertible algebra A is of rank r. If R is an integral domain, then any invertible R-algebra is an integral domain of finite rank.

Theorem 4.1. An R-algebra A is invertible of rank one if and only if A is stably equivalent to $S_R(N)$ for some $N \in \mathrm{Pic}(R)$.

Applying [7] to Theorem 2.4 and Theorem 4.1 we have the following

Corollary 4.2. Let A be an invertible R-algebra of rank one such that R_{red} is an F-ring. Then $A \cong_R S_R(N)$ for some $N \in \mathrm{Pic}(R)$.

In a similar way we have the following corollary of Theorem 2.5 and Theorem 4.1 by [7].

Corollary 4.3. Let A and B be any pair of invertible R-algebras of rank one. Suppose R contains an infinite field. Then $A \otimes_R B \cong_R S_R(M) \otimes_R S_R(N)$ for some $M, N \in \mathrm{Pic}(R)$.

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